Generation QI: Picturing Quantum Weirdness

Week 2 - Day 1 and 2
Operational Probabilistic Theories

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ICTQT – University of Gdańsk **August 14th&16th 2023**

Let us warm up by proving some elementary properties of the trivial systems and identity tests/events in operational languages.

Exercise 1. *Show that the trivial system is unique.*

(*Hint:* Just use the defining property of a trivial system.)

Exercise 2. *Show that the identity test, for every system, is unique, and that it is a singleton test. Also, show that this also implies $\mathscr{I}_B \circ \mathscr{T}_x = \mathscr{T}_x = \mathscr{T}_x \circ \mathscr{I}_A$, where $\mathscr{I}_B$ and $\mathscr{I}_A$ are the identity event for system B and A, respectively, while $\mathscr{T}_x$ is an arbitrary event from system A to system B.*

(*Hint:* Use the defining property of an identity test, combined with the fact that the singleton set plays here the role of a unit with respect to the Cartesian product; in other words, if X and Y are outcome spaces, the following holds: $X \times Y = Y \iff X = \{*\}$.)

Now, let us briefly discuss about *reversible* tests and events. We say a test $\mathsf{R}_X^{A\to B} \in \mathsf{RevTest}(A \to B) \subseteq \mathsf{Test}(A \to B)$ if $\exists\ (\mathsf{R}_l^{-1})_Y^{B\to A}, (\mathsf{R}_r^{-1})_Z^{B\to A} \in \mathsf{Test}(B \to A)$ such that

$$(\mathsf{R}_l^{-1})_X^{B\to A} \circ \mathsf{R}_X^{A\to B} = \mathsf{I}_*^{A\to A} \tag{1}$$

and

$$\mathsf{R}_X^{A\to B} \circ (\mathsf{R}_r^{-1})_Z^{B\to A} = \mathsf{I}_*^{B\to B} \tag{2}$$

Now, $(\mathsf{R}_l^{-1})_X^{B\to A}$ and $(\mathsf{R}_r^{-1})_X^{B\to A}$ are called the *left*- and the *right- inverse* of $\mathsf{R}_X^{A\to B}$, respectively. One similarly defines reversible events. Notice that, for an arbitrary test/event, the existence of the left-inverse does not imply the existence of the right-inverse, nor viceversa.

Exercise 3. *Show that the reversible tests are singleton tests. Exhibit the example of a test having a left-inverse (right-inverse) but not a right-inverse (left-inverse).*

Exercise 4. *Show that the left- and right- inverses of reversible tests coincide, i.e. $(\mathsf{R}_l^{-1})_*^{B\to A} =: (\mathsf{R}^{-1})_*^{B\to A} := (\mathsf{R}_r^{-1})_*^{B\to A}$, and that they are unique.*

(*Hint:* Use the definition of reversible test, combined with the associativity property of the sequential composition of tests.)

Exercise 5. *Show that reversible tests are closed under both sequential and parallel compositions, namely:*

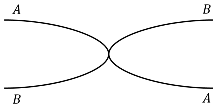
$$\forall\ \mathsf{R}_*^{A\to B} \in \mathsf{RevTest}(A \to B), \forall\ \mathsf{R}_*^{B\to C} \in \mathsf{RevTest}(B \to C), \quad \mathsf{R}_*^{B\to C} \circ \mathsf{R}_*^{A\to B} \in \mathsf{RevTest}(A \to C) \tag{3}$$

and

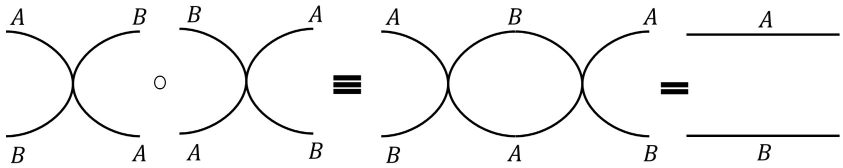
$$\forall\ \mathsf{R}_*^{A\to B} \in \mathsf{RevTest}(A \to B), \forall\ \mathsf{R}_*^{C\to D} \in \mathsf{RevTest}(C \to D), \quad \mathsf{R}_*^{A\to B} \boxtimes \mathsf{R}_*^{C\to D} \in \mathsf{RevTest}(AC \to BD). \tag{4}$$

Exercise 6. *Show that the set of reversible tests $\mathsf{RevTest}(S \to S)$, for every system S, forms a group under sequential composition $\circ$.*

In an operational language Ω, the following feature is often required:

$\forall A, B \in \mathsf{Sys}(\Omega)\ \exists$ 

$\in \mathsf{RevTest}(AB \to BA)$ such that



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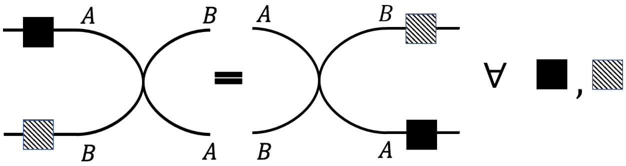

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The above process is known as the *swap*.

Exercise 7. *Consider an operational language Ω.*
(a) Using the properties of the trivial system $I \in \mathsf{Sys}(\Omega)$, along with those of sequential and parallel composition, show, just employing the symbolic notation*, that:*

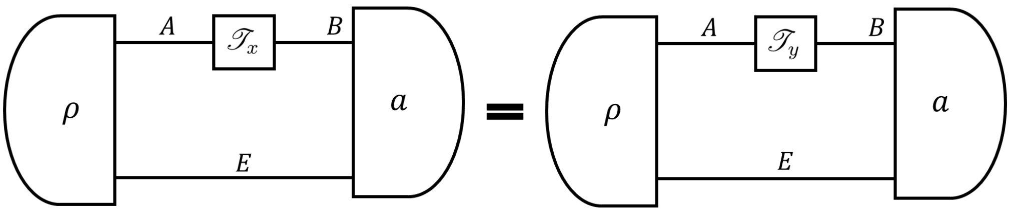
$$P_X \boxtimes Q_Y = Q_Y \circ P_X = Q_Y \boxtimes P_X = P_X \circ Q_Y \tag{5}$$

and likewise

$$\mathscr{P}_x \boxtimes \mathscr{Q}_y = \mathscr{Q}_y \circ \mathscr{P}_x = \mathscr{Q}_y \boxtimes \mathscr{P}_x = \mathscr{P}_x \circ \mathscr{Q}_y \tag{6}$$

hold in Ω for all $P_X, Q_Y \in \mathsf{Test}(I \to I)$ and all $\mathscr{P}_x, \mathscr{Q}_y \in \mathsf{Event}(I \to I)$. Then, proceed to write these properties in diagrammatic form.
(b) Assuming that Ω is also a probabilistic model, and that $\boxtimes$ is, for $\mathsf{Events}(I \to I)$, the usual multiplication of real numbers, show that the disconnected components of a closed diagram (i.e. of a test in $\mathsf{Test}(I \to I)$) are associated with statistically independent (a.k.a. uncorrelated) tests.

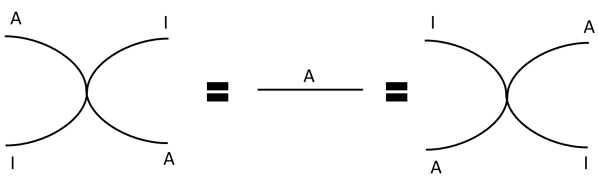
Now, for probabilistic models, one can define the notion of *probabilistic equivalence*. We say that two events $\mathscr{T}_x, \mathscr{T}_y \in \mathsf{Event}(A \to B)$ are probabilistically equivalent, denoted by $\mathscr{T}_x \sim \mathscr{T}_y$, if:



holds $\forall \rho \in \mathsf{Event}(I \to AE)$ and $\forall a \in \mathsf{Event}(BE \to I)$.

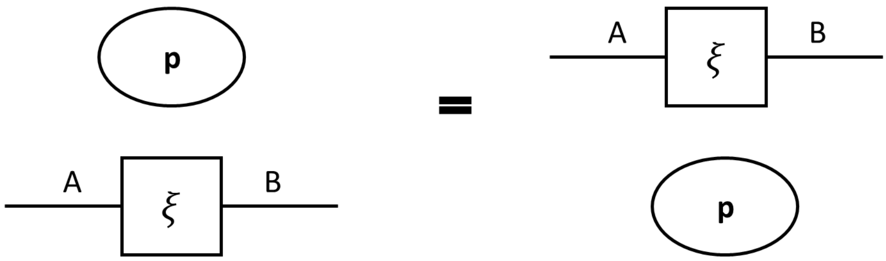
Exercise 8. *Show that probabilistic equivalence is indeed an equivalence relation.*

Exercise 9. *Consider an operational language Ω.*
(a) Using the following property of the swap:

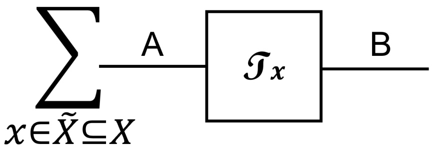


$\forall A \in \mathrm{Sys}(\Omega)$,

combined with another known property of the swap, show that:



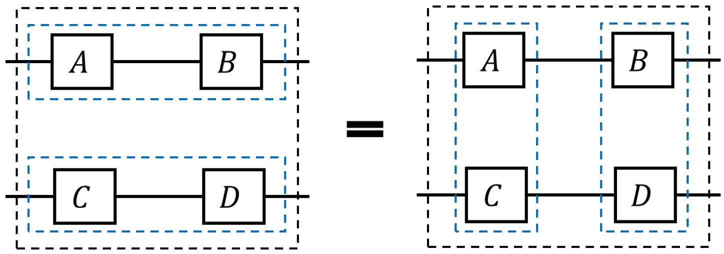
holds for all scalar events $p \in \mathsf{Event}(I \to I)$*, systems* $A, B \in \mathsf{Sys}(\Omega)$*, and events* $\xi \in \mathsf{Events}(A \to B)$*.*
(b) Assuming that Ω *is also a quotient probabilistic model (OPT), show that every sum of the form:*



$$, \quad T_X^{A\to B} = \{\mathscr{T}_x\}_{x\in X} \in \mathsf{Test}\,(\Omega)$$

corresponds to the "partial marginalisation" (a.k.a. coarse-graining), over the outcomes in $\tilde{X}$*, of every probability distribution that can be associated with* $T_X^{A\to B}$ *in the OPT* Ω*. In the case where* $\tilde{X} = X$*, the above sum precisely corresponds with* the marginal distributions *on* X*.*

Now, let us briefly discuss the following *exchange law*:



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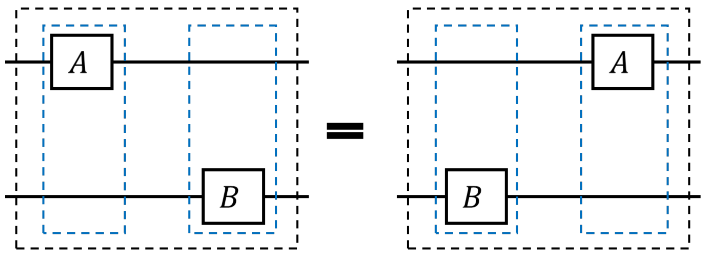
From the exchange law one derives the following identity:



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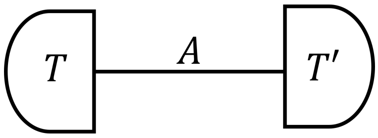
The above identity allows one to make sense of what "locality" means at an operational level. This identity is another sliding property similar to the one holding for the swap. Now, thanks to the exchange law and to the existence of the identity processes, it is possible to stretch wires at will in OPT diagrams.

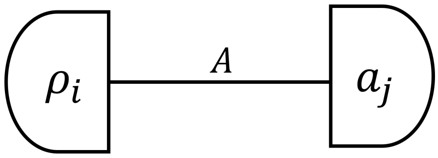
Exercise 10. *Verify that the above exchange law (and the sliding property accordingly) holds in quantum theory.*

Let us now briefly discuss the so-called *causality principle* for OPTs. According to this principle, the probability associated to every preparation-event is independent of the choice of the observation-test. Consider a so-called *prepare-and-measure* experiment involving a preparation-test $T = \{\rho_i\}_{i\in X} \subset \mathsf{St}(A)$ and an observation-test $T' = \{a_j\}_{j\in Y} \subset \mathsf{Eff}(A)$:



Then, the joint probability of the preparation-event ρ_i and the observation-event a_j is given as follows:

$$p(i,j|T,T') \quad \equiv$$



In general, the marginal probability distribution $\sum_{j\in Y} p(i,j|T,T') \equiv p(i|T,T')$ will depend on the choice of observation-test T'. However, if the causality principle holds, one has that:

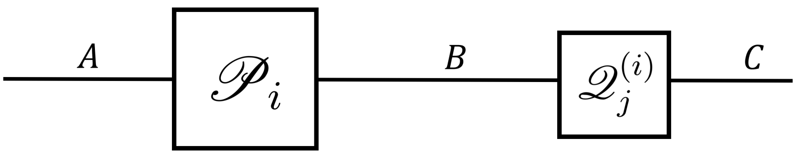
$$p(i|T,T') = p(i|T) =: p(i). \tag{7}$$

An OPT where the causality principle holds is called *causal*.

Exercise 11. *Show that an OPT Θ is causal if and only if there is a unique deterministic effect for every system in* $\mathsf{Sys}(\Theta)$.

(*Hint:* On the one hand, using probabilistic equivalence, show that if the probability associated with every state is independent of the observation-test, then any two deterministic effects must coincide. On the other hand, assuming the uniqueness of the deterministic effect for every system, show that the probability associated with every state, being a marginal probability, is equal for every choice of observation-test.)

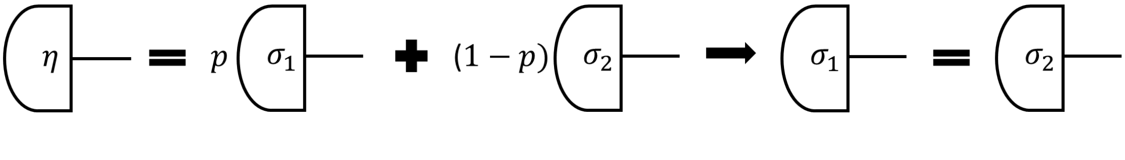
Let us now discuss *conditional tests*. In a causal OPT, the choice of performing given tests can be conditioned upon the occurrence of the outcomes of a preceding test. Consider the test $P_X^{A\to B} = \{\mathscr{P}_i\}_{i\in X}$, and the family of tests $Q^{(i)}{}_Y^{B\to C} = \{\mathscr{Q}_j^{(i)}\}_{j\in Y_i}$ for each value of $i \in X$. A conditioned test is a test $T_Z^{A\to C} := \{\mathscr{Q}_j^{(i)} \circ \mathscr{P}_i\}_{(i,j)\in Z}$ where $Z := \bigcup_i \{\{i\} \times Y_i\}$. One can represent the events of the conditional tests in the following way:



Exercise 12. *Given an OPT Θ, show that if all the conditional tests are allowed in Θ, then Θ is causal.*

(*Hint:* As per **Exercise 11**, it suffices to show the uniqueness of the deterministic effect for every system in the OPT Θ. In order to do so, just consider conditional prepare-and-measure tests, and use the fact that probability distributions must be normalised.)

Next, we will discuss the notion of *pure states* in an OPT. A pure state can be defined as a deterministic state $\eta \in \mathsf{St}(\Theta)$ such that for $p \in (0,1)$ and $\sigma_1, \sigma_2 \in \mathsf{St}(\Theta)$ one has the following implication:

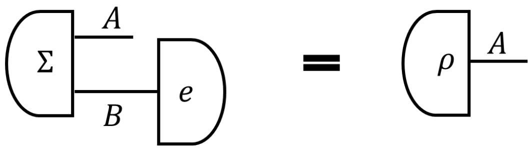


The above condition, in convex analysis, is called *extremality.*

Exercise 13. *Prove that reversible transformations map pure states to pure states, i.e. they preserve purity of states.*

In quantum theory, pure states are given by rank-1 orthogonal projectors.

Exercise 14. ***(Purification principle and Schrödinger–HJW theorem)***
Prove that for every state $\rho \in \mathsf{St}(A)$ in quantum theory (QT) there exists a system $B \in \mathsf{Sys}(QT)$ and a pure state $\Sigma \in \mathsf{St}_1(AB)$, called the purification *of the state ρ, such that:*



where e is the unique deterministic effect of every quantum system (namely, the partial trace). What can be said about the separability of a purification of a mixed state?
(*Hint:* Diagonalise an arbitrary, and generally mixed, quantum state ρ, and use the eigenvalues and the corresponding eigenstates to come up with a desired bipartite pure state, being a purification of ρ. Notice that the exercise can be rephrased as: *"Try to add your name to a result that has been independently re-discovered from time to time by a bunch of physicists over the last century".*)

Exercise 15. *Prove that, in a finite-dimensional simplicial theory (state spaces are d-simplices with $d \in \mathbb{N}$) the reversible tests from system A to itself amount just to all the permutations of those states of A corresponding to the vertices of the simplices.*
(*Hint:* Use the result of **Exercise 13**.)

In the case of finite-dimensional quantum theory, the reversible tests are in one-to-one correspondence with the unitary channels, usually denoted by U. The Krauss decomposition of such a test has only one element, which is the associated deterministic event $U \cdot U^{\dagger}$.

Generation QI: Picturing Quantum Weirdness

Week 2 - Day 3

Spekken's Toy Theory

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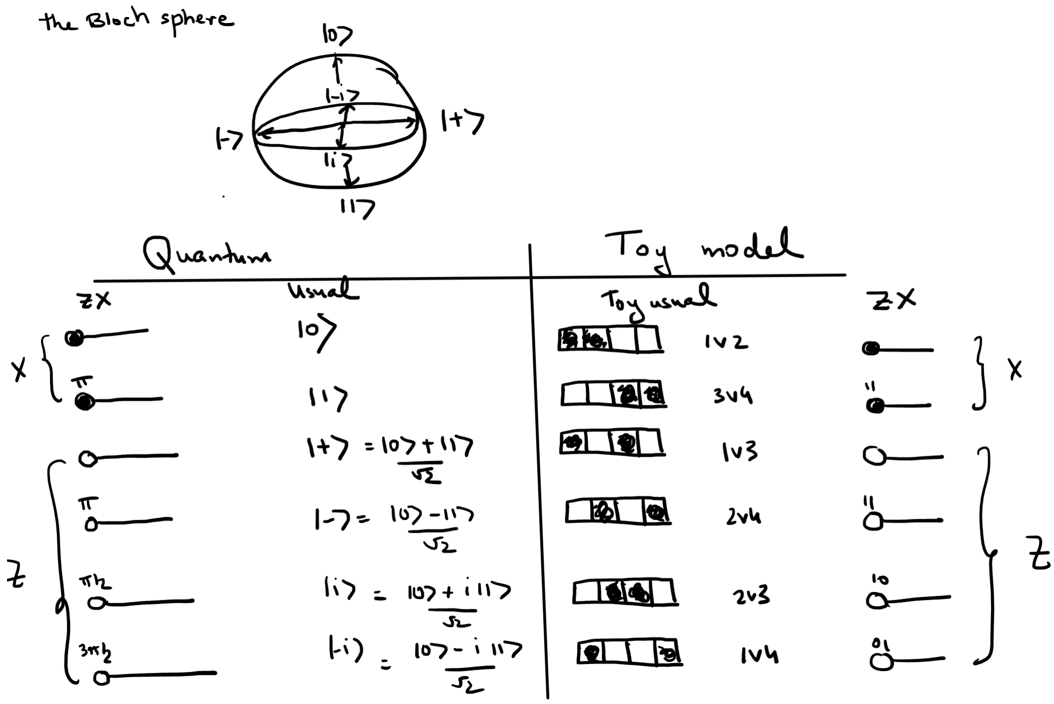
Tutorials: Vinicius Pretti Rossi, Sina Soltani, Beata Zjawin

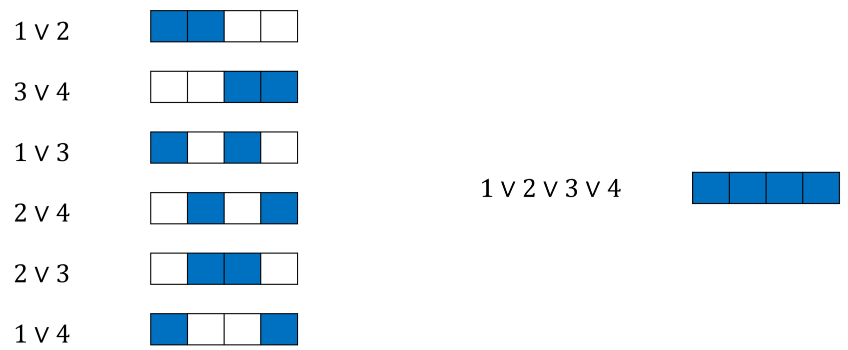
ICTQT – University of Gdańsk **August 17th 2023**

Exercises

Here is an important table useful for these exercises.



Let us briefly discuss Spekken's toy theory. The simplest possible system in the toy theory, called the *elementary system* has four ontic states denoted by $1, 2, 3$ and 4. The epistemic states or the state of knowledge for which the balance of knowledge principle holds are the ones for which answer to maximum one canonical question can be known.



States of maximal knowledge States of non-maximal knowledge

Two states are said to be *compatible* if the intersection of their ontic bases is the ontic base of a valid epistemic state.

Exercise 1: *Check whether the following set of states are compatible:*

(i) $2 \vee 4$ and $1 \vee 2 \vee 3 \vee 4$

(ii) $1 \vee 4$ and $1 \vee 3$

(iii) $1 \vee 2$ and $3 \vee 4$

Convex combination in the toy theory is analogous to incoherent superposition in quantum theory. One can find the convex combination if two states are disjoint and if the union of their ontic bases forms an ontic base of a valid epistemic state.

Exercise 2: *For the following states, find the states after the convex combination if it exists.*

(i) $1 \vee 4$ and $3 \vee 4$

(ii) $2 \vee 4$ and $1 \vee 3$

(iii) $1 \vee 2$ and $1 \vee 2 \vee 3 \vee 4$

One can draw the following analogy between a single elementary system in toy theory and the system described by a qubit — the two-dimensional Hilbert space in quantum theory.

$$
\begin{aligned}
1 \vee 2 &\iff |0\rangle \\
3 \vee 4 &\iff |1\rangle \\
1 \vee 3 &\iff |+\rangle \\
2 \vee 4 &\iff |-\rangle \\
2 \vee 3 &\iff |+i\rangle \\
1 \vee 4 &\iff |-i\rangle \\
1 \vee 2 \vee 3 \vee 4 &\iff \mathbf{I}/2
\end{aligned}
$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ and $|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i\,|1\rangle)$ and $\mathbf{I}/2 = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ is the maximally mixed state for qubits. For the pure states in the elementary system of toy theory, one can define coherent superposition through four binary operations. For two states $a \vee b$ and $c \vee d$ ($a < b, c < d, a \neq b, c \neq d$, and $a, b \in \{1, 2, 3, 4\}$) the four operations are defined as follows:

$$
\begin{aligned}
(\mathrm{a} \vee b) +_1 (c \vee d) &= a \vee c \\
(\mathrm{a} \vee b) +_2 (c \vee d) &= b \vee d \\
(\mathrm{a} \vee b) +_3 (c \vee d) &= b \vee c \\
(\mathrm{a} \vee b) +_4 (c \vee d) &= a \vee d
\end{aligned}
$$

The above operations can be considered commutative if $(a \vee b) +_i (c \vee d) = (c \vee d) +_i (a \vee b)$ for some $i \in \{1, 2, 3, 4\}$.

Exercise 3: *Comment on the commutativity property of the above four operations.*

Solution 3: You can verify that $+_1$ and $+_2$ are indeed commutative. On the other hand $+_3$ and $+_4$ are not. For example $(1 \vee 2) +_4 (3 \vee 4) = 1 \vee 4$ but $(3 \vee 4) +_4 (1 \vee 2) = 3 \vee 2 = 2 \vee 3$ and $1 \vee 4 \neq 2 \vee 3$

Now let us discuss a bit about universal state inverter. For pure states, one can define a state inverter as a transformation that deterministically maps the state to its orthogonal state. In quantum theory, the action of a state inverter would be to map a state $|\psi\rangle$ to its orthogonal state $|\psi^{\perp}\rangle$

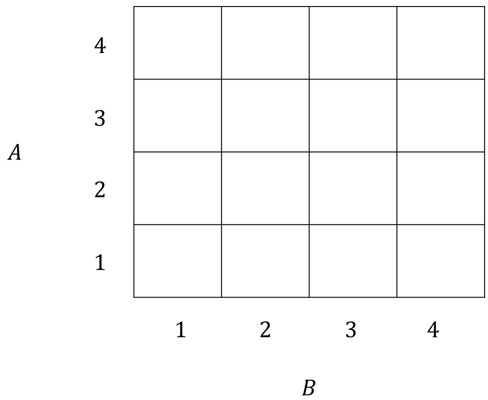
Exercise 4: *Show that there is no universal state inverter for quantum pure states.*

Hint: Check for qubits and use the linear property of unitary transformations. Alternately, use *ZX calculus* to come up with a proof of the same.

Exercise 5: *Show that there is no universal state inverter for elementary systems while considering the toy theory.*

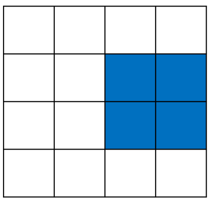
Hint: Notice that transformation in toy theory is a permutation of ontic states. Alternately, use *ZX calculus* to find the proof.

Now consider the simplest composite system, i.e., a pair of elementary systems in toy theory. The ontic state of the composite system is represented as $i \cdot j$ where $i, j \in \{1, 2, 3, 4\}$ are the ontic states of two individual sub-systems, say A and B. This can be represented as a 4×4 array.



According to the knowledge balance principle, only two of the four canonical questions may be answered in the state of maximum knowledge.

Exercise 6: *Among the following, which ones represent some valid states for the composite system. Also, identify if they correspond to state of maximal (or non-maximal) knowledge for individual subsystems. Among the valid states (maximal or non-maximal knowledge states) comment if they are analogous to entangled states or otherwise.*



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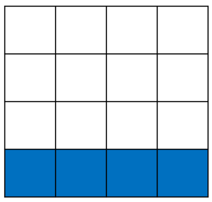

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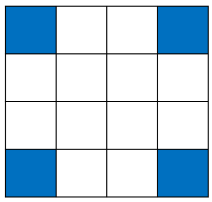

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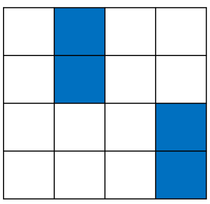

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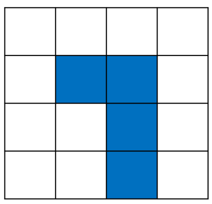

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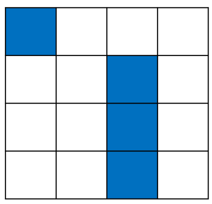

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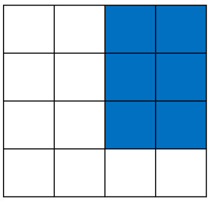

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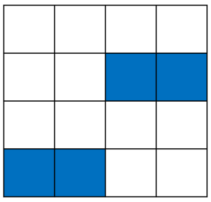

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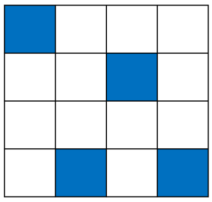
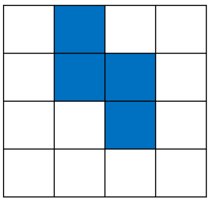


(i)



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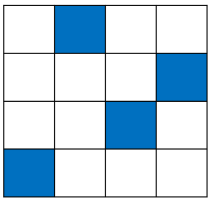

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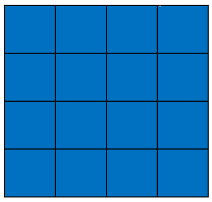

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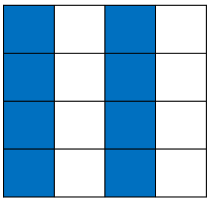

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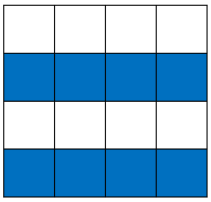

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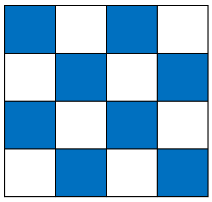


(o)

In quantum mechanics, let us discuss the notion of cloning a pure state. Given any pure state $|\psi\rangle$ and any fixed state $|\gamma\rangle$, the process of cloning, deterministically transforms this state of the composite system in the following manner:

$$|\psi\rangle \otimes |\gamma\rangle \longrightarrow |\psi\rangle \otimes |\psi\rangle \qquad \forall\, |\psi\rangle$$

Exercise 7: *Prove that there cannot be a universal unitary transformation that clones all pure quantum states in the manner described above.*

Hint: Use the properties of unitary transformation. Also, notice that the state of the second subsystem is always a fixed state. Alternately, use *ZX calculus* for the proof.

Exercise 8: *Show that such cloning of states is not possible in elementary toy theory.*

Hint: Start with a composite system where the state of the first system is supposed to be cloned and the initial state of the second system is some fixed state say $(2 \vee 3)$. Now, use the properties of transformations in the toy theory. Alternately, you can also use *ZX calculus* for the proof. Alternately, you can also use *ZX calculus* for the proof.

Now, consider two parties Alice and Bob where Alice can communicate only one qubit to Bob. It was shown by Holevo that by sending only one qubit, Alice can communicate to Bob only one classical bit of information. This imposes a restriction on the maximum amount of information that can be communicated using quantum systems. However, if Alice and Bob have already shared entanglement between them then by sending just one qubit, Alice can communicate two classical bit of information to Bob. This is known as dense coding. Say, Alice and Bob share the maximally entangled state $|\Phi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Conditioned upon the four messages $00, 01, 10$ or 11, Alice applies one of the four unitary transformations $\mathbb{1}, \sigma_Z, \sigma_X$ and $i\sigma_Y$ on the first subsystem. Under this action, the state $|\Phi^+\rangle$ evolves to $|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle$ and $|\Psi^-\rangle$ (Bell basis) respectively where $|\Phi^\pm\rangle := \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), |\Psi^\pm\rangle := \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. Next, Alice sends the first subsystem to Bob and then Bob performs a measurement in the Bell basis and get the deterministic outcome (depending on Alice's unitary). Thus, Bob will exactly identify the Bell state and know the message that Alice intended to send. Using this protocol Alice can communicate 2 classical bits of information. This possibility, one may presume, to be exclusive feature of quantum theory. However, as we will see during the next exercise, this feature is also shown by Spekken's toy theory.

Exercise 9: *Show that it is possible to perform dense coding if Alice and Bob have already shared the state* $(1 \cdot 1) \vee (2 \cdot 2) \vee (3 \cdot 3) \vee (4 \cdot 4)$ *and afterwards Alice is allowed to send only one elementary system to Bob.*

Hint: Think of states and transformations in toy theory that are analogous to the ones used in quantum strategy discussed above. Also, find the measurement analogous to bell basis measurement for the toy theory.

Now, we shall briefly discuss something that quantum theory can do, but the toy theory cannot. This interests us since these behaviours get a genuinely non-classical status. Peres-Mermin square is one such proof of non-classicality of quantum theory. Consider a table of measurements over a two-qubit state, with the form as shown below:

$\mathbb{1} \otimes \sigma_Z$	$\sigma_Z \otimes \mathbb{1}$	$\sigma_Z \otimes \sigma_Z$
$\sigma_X \otimes \mathbb{1}$	$\mathbb{1} \otimes \sigma_X$	$\sigma_X \otimes \sigma_X$
$-\sigma_X \otimes \sigma_Z$	$-\sigma_Z \otimes \sigma_X$	$\sigma_Y \otimes \sigma_Y$

.

Observable in each row and column of this matrix multiplies to $\mathbb{1} \otimes \mathbb{1}$ and $-\mathbb{1} \otimes \mathbb{1}$ respectively. Now, assuming that each of these observable has some definite outcome ± 1 assigned to it in a non-contextual manner, it is impossible to find assignments such that the outcomes for observable in each column and row individually multiplies to 1 and -1 respectively.

First, let us see if there are measurements corresponding to Pauli measurements in the toy theory. In ZX calculus, the Pauli-Z measurement is represented by a node with phase π in the Z basis, while a Pauli-X is represented by the same node and phase, but in the X basis.

Exercise 10: *How would you represent the 3 Pauli measurements in the ZX notation of the toy theory?*

Hint: remember that a phase π in the conventional ZX notation is replaced by a phase 11 in the toy theory convention!

Once we settled this up, we are able to show that the toy theory *cannot* yield a Peres-Mermin-like contradiction.

Exercise 11: *Using the analogue of Pauli measurements for the Spekkens' toy theory check that the entries in the Peres-Mermin square cannot be assigned without contradiction.*

Generation QI: Picturing Quantum Weirdness

Week 2 - Day 5
Resource Theory

Lecturers: John Selby, David Schmid
Tutorials: Vinicius Pretti Rossi, Chithra Raj, Beata Zjawin

ICTQT – University of Gdańsk **August 19th 2023**

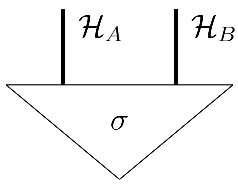
Exercises

In this sheet we're going to work towards quantifying the kind of nonclassicality that we saw in the previous exercise sheet, and we're going to do so by trying to construct a resource theory for it as introduced in the lectures. This means that we need to decide i) what are the set resources, R, that we're interested in here, and ii) how do we know when one resource can be freely transformed into another, $r_1 \succ r_2$. In particular, we're going to be following the approach of [arXiv:1909.04065].

To get there, lets take a step back and think about this "common cause" scenario that has been mentioned a few times now. The basic idea here, is that we have two parties, Alice and Bob, sitting in separate labs, who have no direct way to influence one another. They might, however, have some systems in their lab which could have been interacting with one another at some point in the past when Alice and Bob met up with one another. It is these systems that are the *common cause* which can lead, for example, to correlations between what they observe in their labs. With this in mind, given that we are trying to understand nonclassicality, it is natural to divide the things that Alice and Bob can do into free and nonfree by saying that the transformations they can do freely are those that rely only on a classical common cause – that is, some shared randomness – and the things that they can do nonfreely are those that rely on a quantum common cause – that is, some shared entangled state.

This leaves open the question as to, what are the resources that we wish to consider. Ultimately we want quite a general notion of a resource, but we'll build up to that by considering special cases.

To begin, lets consider the resources to be bipartite quantum states shared between Alice and Bob. Using a diagrammatic notation we draw these as:



(1)

where $\mathcal{H}_A$ is the Hilbert space associated to the quantum system in Alice's lab and $\mathcal{H}_B$ the Hilbert space associated to the quantum system in Bob's lab.

Exercise 1: *Taking the set of resources to be the set of bipartite quantum states, when can one resource be freely transformed into another in the common cause scenario?*

Now, lets do a basic sanity check on what you've come up with in the previous exercise.

Exercise 2: *Show that this defines a preorder on the set of bipartite quantum states, namely that $r \succ r$ and that if $r_1 \succ r_2$ and $r_2 \succ r_3$ implies $r_1 \succ r_3$.*

An important notion in a resource theory is that of a *free resource*, that is, a resource that can be freely created from no resource.

Exercise 3: *Show that the resources that can be freely created from nothing are separable bipartite quantum states.*

Another important notion in a resource theory is whether or not we can freely *discard* resources (n.b. this is not always the case, e.g., nuclear waste).

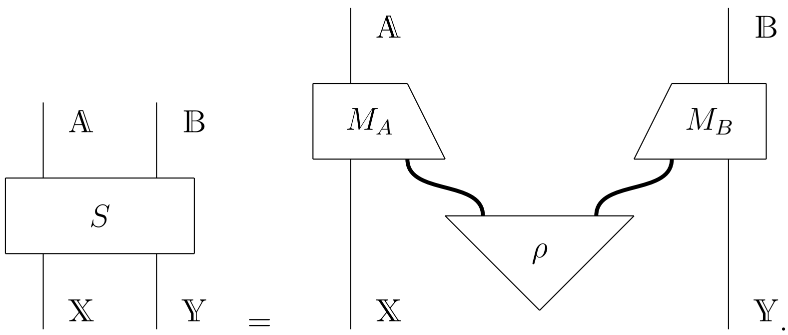
Exercise 4: *Show that we can always discard any resource and end up with no resource.*

Putting these last two exercises together, and using one of the basic properties of a preorder, we can...

Exercise 5: *Show that whatever resource we start with, we can always freely convert it into any separable state.*

What we have here is what is known as the resource theory of local operations and shared randomness (LOSR) for bipartite quantum states. It is one way of quantifying the entanglement in such states. It is worth flagging up the fact that there are other resource theories that one can consider in order to quantify the entanglement in such states, and which one you should use ultimately depends on the scenario that you are considering. Another commonly considered option is the resource theory of local operations and classical communication (LOCC) – that is, Alice and Bob can now pick up the telephone and chat to each other, but cannot send each other quantum systems. Whether this classical communication is something that should be thought of as being free depends on the situation, but, for example, if they are space-like separated from one another then such communication definitely should not be free, as they would need to have some device that violates the laws of relativity in order to do so!

Lets now turn to another kind of resource that is commonly considered, namely, bipartite stochastic maps which are quantum realisable in the Bell scenario. That is, stochastic maps S such that there exists ρ, M_A and M_B such that



(2)

Exercise 6: *Taking the set of resources to be the set of quantum realisable bipartite stochastic maps, when can one resource be freely transformed into another in the common cause scenario?*

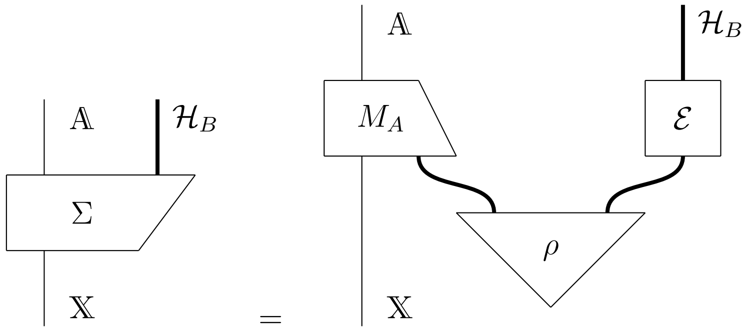
Again we can perform the basic sanity check that this does indeed define a valid preorder, do this as a bonus exercise if you want! More interesting, however, is to look at the set of free resources.

Exercise 7: *What are the free resources, that is, what bipartite stochastic maps can we create from nothing?*

Similarly, we can argue that we can always discard any of these resources for free (feel free to show this as another bonus exercise), and hence given any resource we can always freely convert it into one of these free resources.

What we have here is what is known as the resource theory of local operations and shared randomness (LOSR) for bipartite stochastic maps. It is one way of quantifying the nonclassicality of such stochastic maps. One thing to think about is, what would happen if we tried to use LOCC here instead?

As our final example, lets consider the resources to be assemblages which are quantum realisable in the EPR scenario. That is, assemblages Σ such that there exist ρ, M_A and $\mathcal{E}$ such that:



(3)

Exercise 8: *Taking the set of resources to be the set of quantum realisable assemblages in the EPR scenario, when can one resource be freely transformed into another?*

Again as a bonus question you could show that this does indeed define a valid preorder on quantum realisable assemblages, but again, more interesting is to look at the set of free resources.

Exercise 9: *What are the free resources, that is, what assemblages can we create from nothing?*

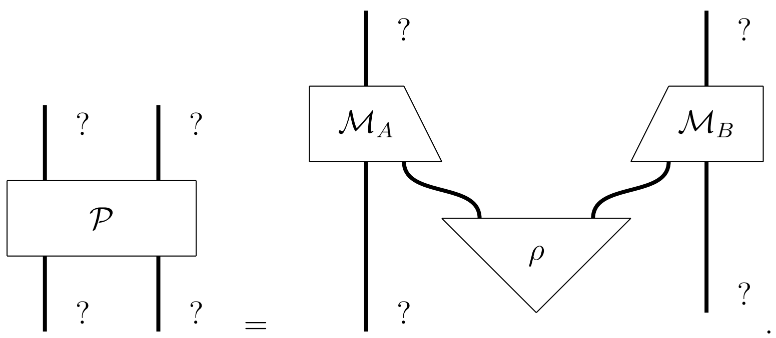
And, again, we can argue that discarding resources is free (which you could also take as a bonus exercise), which means that any resource can always be freely converted into a free resource.

What we have here is what is known as the resource theory of local operations and shared randomness (LOSR) for EPR assemblages. It is one way of quantifying the nonclassicality of such assemblages.

So, by now you have seen how various sorts of nonclassicality, namely, entanglement, Bell nonclassicality, and EPR nonclassicality, can each be quantified as a resource theory in which the free transformations are given by local operations and shared randomness – all that changes is the nature of the resources, and, hence, the nature of the local operations that Alice and Bob can perform.

We can now see how these apparently different notions of nonclassicality can all be unified by taking a 'type-independent' perspective, whereby these three notions of nonclassicality fall out as special cases. That is, we can see how we can put the three resources theories that we have defined so far together into a single unified resource theory.

In particular, we do so by taking the resources to be quantumly realisable bipartite processes $\mathcal{P}$, that are quantumly realisable in a common cause scenario. That is, there exist processes $\mathcal{M}_A$, $\mathcal{M}_B$ and bipartite quantum state ρ such that:



(4)

Exercise 10: *Taking these quantumly realisable bipartite processes in the common cause scenario as resources, when can one resource be freely transformed into another?*

Exercise 11: *What are the free resources, that is, what bipartite processes can we create from nothing?*

This means that we have a resource theory of nonclassicality for type-independent bipartite processes in the common cause scenario. However, in order to really argue that we have unified the previous three resource theories we need to understand how each of them can be recovered from this one.

Exercise 12: *Show that if start with the preorder defined above over all bipartite processes, and then consider the preorder that this defines i) on bipartite quantum states, ii) on bipartite stochastic maps, and iii) on EPR assembalges, that this is the same preorder as we obtained in exercises 1, 6, and 8 respectively.*

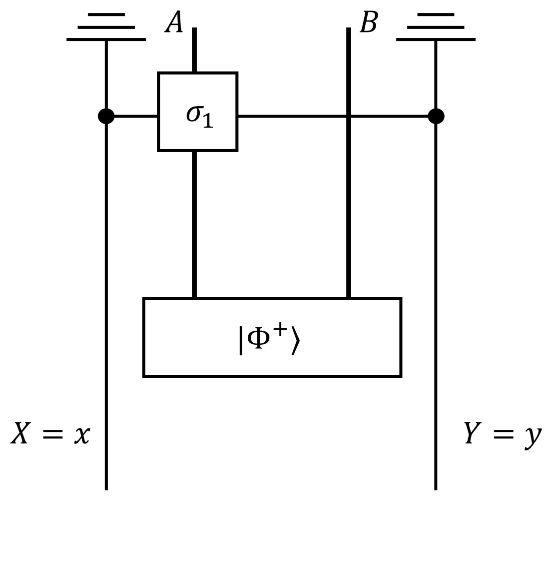
The point of this unification, however, isn't simply so that we can obtain the previous examples by restricting to particular subsets of resources. What it allows, is a) to easily define resource theories of other special cases of interest without having to reinvent the wheel each time, and b) to allow us to quantitatively study the relationship between these apparently different kinds of nonclassicality.

To give an example of the latter point, traditionally entanglement and Bell nonclassicality have been treated as independent notions of nonclassicality. The former traditionally quantified by taking free transformations as local operations and classical communication, and the latter quantified by local operations and shared randomness. The lack of a unified approach led to so-called 'annomolies of nonlocality' wherein apparently less entangled states lead to more nonlocality than apparently more entangled states. Such issues are automatically resolved by taking a unified type-independent approach such as the one you have seen above. See [arXiv:2004.09194] for details of this issue. Note that one could also resolve these issues by developing a type independent approach to the resource theory of local operations and classical communication, the problem here, however, is that in such a resource theory the nonclassicality in Bell scenarios and EPR scenarios trivialises, i.e., every bipartite stochastic map and every EPR assemblage is free, hence the resource theory of local operations and shared randomness that you've seen here seems more interesting to me at least!

Now, considered the situation as described in equation (2). In this scenario, there exists bipartite stochastic maps that are not quantum realisable. That is, there do not exist any ρ, M_A and M_B such that these stochastic maps can be realised. An example of this being the Popescu-Rohrlich (PR)-box. It is completely specified by the following conditional probability distribution.

$$p(ab|xy) = \frac{1}{2}\ \delta_{a\oplus b, x\cdot y} \tag{5}$$

where $a, b, x, y \in \{0, 1\}$. This correlation achieves the algebraic maximum violation, i.e. 4, of the (CHSH) inequality while the quantum maximum is $2\sqrt{2}$.
Let us consider now another process called PHHH ensemble preparation. One realization of this process (which involves signaling between the parties, and hence only works if the inputs and outputs on each side are not at space-like separated) is as shown below.

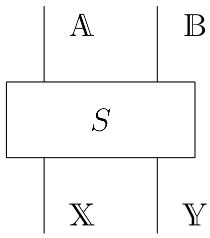


It outputs the Bells state $|\Phi^+\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$ if the product of the inputs $x \cdot y = 0$ and outputs the Bells state $|\Psi^+\rangle = \frac{1}{2}(|01\rangle + |10\rangle)$ if the product of the inputs $x \cdot y = 1$ where $x, y \in \{0, 1\}$.

Exercise 13: *Show that one can obtain PR-box when starting from PHHH ensemble preparation. Also, show that the reverse transformation is also possible if Alice and Bob can share all quantum common cause.*

Hint: Assume Alice and Bob share the Bell state $|\Phi^+\rangle$ for the second part.

Now, let us consider that you start from any resource of as described below



(6)

During the lectures, you have seen that when converting resources of this box type into quantum states using only LOSR transformations, one always obtains free (separable) states. Similarly, one can consider the scenario when you have initially any non-classical assemblage (quantum or even post quantum).

Exercise 14: *Show that when converting resources of this type (any non-classical assemblage) into quantum states using only LOSR transformations, one always obtains free (separable) states.*

Hint: The proof is similar to the previous case discussed during the lectures.

Generation QI: Picturing Quantum Weirdness

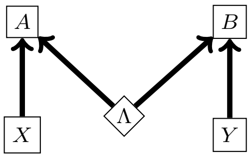
Week 2 - Day 4
Signature of Non-Classicality

Lecturers: Ana Belén Sainz, David Schmid
Tutorials: Vinicius Pretti Rossi, Sina Soltani, Beata Zjawin

ICTQT – University of Gdańsk **August 18th 2023**

Exercises

In today's class you were introduced to the causal perspective on Bell's theorem. One of the key benefits of this perspective is that it very naturally leads to generalisations. In a nutshell, Bell's theorem tells us that if we consider the DAG



(1)

that there are distributions $\mathbf{P}_{\mathbb{AB}|\mathbb{XY}}$ which are realisable with a shared quantum system in place of Λ but which are not realisable with a classical causal model on this DAG. To notate this, we will say that the set of possible quantum correlations for this DAG is called $\mathcal{Q}_{\text{Bell}}$ and the set of possible correlations in a classical causal model is called $\mathcal{C}_{\text{Bell}}$, succinctly then Bell's theorem is saying that $\mathcal{C}_{\text{Bell}} \subset \mathcal{Q}_{\text{Bell}}$.

To be a bit more explicit, recall that the correlations realisable classically are

$$\mathcal{C}_{\text{Bell}} = \{\mathbf{P}_{\mathbb{AB}|\mathbb{XY}} | p(ab|xy) = \sum_{\lambda\in\Lambda} p(\lambda)p(a|x\lambda)p(b|y\lambda),\ \forall p(ab|xy) \in \mathbf{P}_{\mathbb{AB}|\mathbb{XY}}\}. \tag{2}$$

Exercise 1: *Show that the correlators factorise for any $\mathbf{P}_{\mathbb{AB}|\mathbb{XY}}$, i.e., show that*

$$E_{xy} = \sum_{\lambda\in\Lambda} p(\lambda)E_x(\lambda)E_y(\lambda), \tag{3}$$

for some $E_x(\lambda)$ and $E_y(\lambda)$. What are the maximum and minimum values that E_x^λ and E_y^λ may have?

With this in hands, you can immediately get the CHSH classical bound.

Exercise 2: *Show that for any $\mathbf{P}_{\mathbb{AB}|\mathbb{XY}} \in \mathcal{C}_{Bell}$,*

$$|E_{00} + E_{01} + E_{10} - E_{11}| \leq 2. \tag{4}$$

Now you are probably convinced that anything violating this inequality cannot have a classical common-cause explanation. Particularly, some correlations in

$$\mathcal{Q}_{\text{Bell}} = \{\mathbf{P}_{\mathbb{AB}|\mathbb{XY}} | p(ab|xy) = \operatorname{tr}\left\{(M_{a|x} \otimes N_{b|y})\rho\right\},\ \forall p(ab|xy) \in \mathbf{P}_{\mathbb{AB}|\mathbb{XY}}\} \tag{5}$$

can violate the CHSH inequality up to $2\sqrt{2}$. Let us derive this important quantity, called the Tsirielson bound. Consider first that Alice and Bob can perform the following measurements over a qubit:

$$M_{0|0} = N_{0|0} = |0\rangle\!\langle 0|\,, \quad M_{1|0} = N_{1|0} = |1\rangle\!\langle 1|\,; \tag{6}$$

$$M_{0|1} = N_{0|1} = |+\rangle\langle+| \,, \quad M_{1|1} = N_{1|1} = |-\rangle\langle-| \,. \tag{7}$$

Exercise 3: *Show that, when the measurements are defined as in Eqs. (6) and (7), for any state ρ,*

$$E_{xy} = \mathrm{tr}\left\{(\sigma_x \otimes \sigma_y)\rho\right\}, \tag{8}$$

where σ_0 is the Pauli Z operator, and σ_1 is the Pauli X operator. What is the maximum and minimum values E_{xy} can achieve?

This might naïvely give the intuition that the quantum upper bound for this Bell test will be the same as in the classical case. However, consider the operators $\sigma_\pm := \sigma_0 \pm \sigma_1$.

Exercise 4: *What are the eigenvalues of $\sigma_\pm$? Write down $I_{CHSH}(\mathrm{tr}\left\{(M_{a|x} \otimes N_{b|y})\rho\right\})$ as a function of σ_0, σ_1, $\sigma_\pm$ and ρ. What is the maximum value it can achieve?*

This demonstrates that for these particular measurements, there will be a state in which the CHSH functional reaches the value $2\sqrt{2}$. It is left to demonstrate that this is the best we can do with quantum theory. For this consider any operators

$$\hat{\sigma}_x = M'_{0|x} - M'_{1|x}, \tag{9}$$

where $M'_{a|x}$ are projective measurements. Because these are just resolutions of the identity, we can rewrite

$$\hat{\sigma}_x = 2M'_{0|x} - \mathbb{1}, \tag{10}$$

Therefore, if we consider two projectors $M'_{0|0}$ and $M'_{0|1}$ living in the same hemisphere of the Bloch sphere, we have the generalised $\hat{\sigma}_-$ as

$$\hat{\sigma}_- = 2(M'_{0|0} - M'_{0|1}). \tag{11}$$

Exercise 5: *Show that for any qubit projectors $M'_{0|0}$, $M'_{0|1}$, the eigenvalues of $\hat{\sigma}_-$ will be at best $\pm\sqrt{2}$. (Hint: without loss of generality, you can rotate both projectors such that one of them becomes, for instance, $|0\rangle\langle 0|$, and the other is just a generic $|\psi\rangle\langle\psi|$ in the same hemisphere.)*

Of course we are focused on quantum realisations of the CHSH scenario with projective measurements, but it is possible to prove that this bound still holds for POVMs (in fact, we will soon see that any quantum realisation of $\mathbf{P}_{\mathbb{AB}|\mathbb{XY}}$ can be thought of as projective measurements acting over a pure state).

Notice that by the triangle inequality,

$$|E_{00} + E_{01} + E_{10} - E_{11}| \leq |E_{00}| + |E_{01}| + |E_{10}| + |E_{11}|, \tag{12}$$

and since we have already stated that E_{xy} can only reach values between -1 and 1, we will at most have

$$I_{CHSH}(\mathbf{P}_{\mathbb{AB}|\mathbb{XY}}) \leq 4. \tag{13}$$

Consider now the following distribution:

$$\mathbf{P}^{PR}_{\mathbb{AB}|\mathbb{XY}} = \left\{\frac{1}{2}\delta_{a\oplus b, x\cdot y}\right\}_{a\in\mathbb{A}, b\in\mathbb{B}, x\in\mathbb{X}, y\in\mathbb{Y}}, \tag{14}$$

where $\oplus$ is sum modulo 2.

Exercise 6: *Calculate the correlators E_{xy} for the above distribution. Is the CHSH inequality violated by it? What about the Tsirielson bound?*

We see that the given correlation cannot be realised by a quantum Bell scenario, since we have already proved that quantum Bell scenarios do not violate the Tsirielson bound. Is this a nice correlation to explain the Bell DAG at all? For instance, we assume no-signalling in Bell scenarios, so if this correlation allows superluminal signalling, we might want to ignore it at all.

Exercise 7: *Verify that* $\mathbf{P}^{PR}_{\mathbb{AB}|\mathbb{XY}}$ *satisfies the no-signalling condition.*

So there are correlations that satisfy no-signalling, and therefore could in principle provide a faithful model for the Bell DAG, but that cannot be realised quantumly. The one above was first proposed by Popescu and Rorhlich (that's why the superscript PR). These general correlations often receive the name of *no-signalling correlations*, and luckily there is a way of investigating them in a mathematical framework closer from classical theory, thanks to [arXiv1301.2170]. This representation is given by

$$\hat{\mathcal{Q}}_{\mathbf{Bell}} := \{\mathbf{P}_{\mathbb{AB}|\mathbb{XY}}|p(ab|xy) = \sum_{\lambda\in\Lambda} q(\lambda)p(a|x\lambda)p(b|y\lambda),\ \forall p(ab|xy) \in \mathbf{P}_{\mathbb{AB}|\mathbb{XY}}\}, \tag{15}$$

where $q(\lambda)$ is a *quasiprobability distribution.* Such distributions are a relaxation of probability distributions in which we drop the demand of non-negativity. Therefore, $q(\lambda)$ is still normalised, but $q(\lambda) \in \mathbb{R}$ rather than $q(\lambda) \in [0,1]$. We can similarly define quasi conditional distributions as $q(a|b)$ such that $q(a|b)$ is normalised for all values of b, but still can take any real value rather than $[0,1]$. All of the rules that we have been using for manipulating probability distributions immediately carry over to analogous manipulations for quasi probability distributions.

Exercise 6: *Verify that any* $\mathbf{P}_{\mathbb{AB}|\mathbb{XY}} \in \hat{\mathcal{Q}}_{Bell}$ *indeed satisfies the no-signalling condition.*

Investigating the borders between $\mathcal{Q}_{\text{Bell}}$ and $\hat{\mathcal{Q}}_{\text{Bell}}$ is an ongoing problem in quantum foundations, and we will hopefully explore in the lectures and maybe a future problem set some of the approaches for it.

To conclude this tutorial, let us prove that quantum theory does not admit of a noncontextual ontological model. Consider the prepare-and-measure scenario with the following matrices representing both states and effects:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad \sigma_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_2 = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}; \tag{16}$$

$$\sigma_3 = \frac{1}{4}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}; \quad \sigma_4 = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}; \quad \sigma_5 = \frac{1}{4}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}. \tag{17}$$

To each preparation, there will be an epistemic state $\mu(\lambda|k)$, with $k = 0, ..., 5$, and to each measurement outcome there will be a response function $\xi(k|\lambda)$. Notice that the outcomes 0 and 1 belong to the same measurement. The same for 2 and 3, and 4 and 5 respectively. Also, we can assume that $\xi(k|\lambda) \in \{0,1\}$ in this particular case.

Exercise 7: *Which constraints must these epistemic states satisfy to be compatible with the null probabilities in this model?*

Finally, notice that

$$\begin{aligned}
\frac{1}{2}\mathbb{1} &= \frac{1}{2}(\sigma_0 + \sigma_1) && (18)\\
&= \frac{1}{2}(\sigma_2 + \sigma_3) && (19)\\
&= \frac{1}{2}(\sigma_4 + \sigma_5) && (20)\\
&= \frac{1}{3}(\sigma_0 + \sigma_2 + \sigma 4) && (21)\\
&= \frac{1}{3}(\sigma_1 + \sigma_3 + \sigma_5), && (22)
\end{aligned}$$

and the same for the response functions.

Exercise 8: *Which constraints the ontological model must satisfy to accommodate these equivalence classes? Show that this set of constraints is incompatible with the one obtained in the last exercise.*

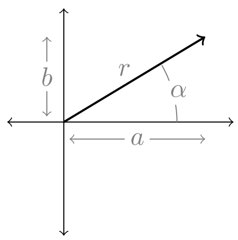
Gdansk Summer School

Day 1, Picturing Quantum Weirdness 2023

Let's make sure we are all on the same page with regards to complex numbers. There are two main ways to write down a complex number. The first is the familiar **cartesian form**: $\lambda = a + ib$ in terms of real numbers $a, b \in \mathbb{R}$ and $i := \sqrt{-1}$. The second, possibly less familiar, is the **polar form**: $\lambda = re^{i\alpha}$ for a positive real number $r \in \mathbb{R}_{\geq 0}$ called the **magnitude** and an angle $\alpha \in [0, 2\pi)$ called the **phase**. Phases play an important role in quantum theory, and they are often where the 'quantum magic' happens, so we'll run into the polar form a lot. These two representations are related by the trigonometric identity:

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \quad \Longrightarrow \quad \begin{cases} a = r\cos(\alpha) \\ b = r\sin(\alpha) \end{cases} \tag{1}$$

which can be visualised in the **complex plane** as follows:

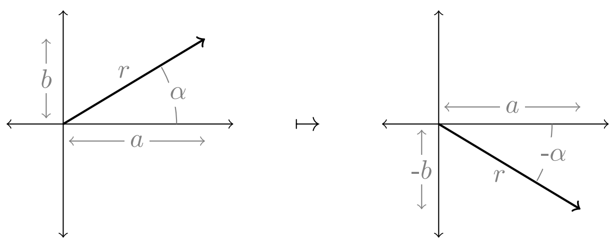


(2)

A useful operation on the complex numbers is the **complex conjugation** $\lambda \mapsto \overline{\lambda}$, which flips the sign of b in cartesian form or the sign of α in polar form:

$$a + ib \mapsto a - ib \qquad\qquad re^{i\alpha} \mapsto re^{-i\alpha}$$

It's pretty straightforward to derive from (1) that these two operations are the same for a complex number λ. This can be visualised by realising that flipping the sign of b or α amounts to a vertical reflection of the complex plane:



We will slightly overload the term 'phase', and also refer to complex numbers of the form $e^{i\alpha}$ as phases. If there is some ambiguity, we will call $e^{i\alpha}$ the phase and α the **phase angle**. Because of the behaviour of exponents, when we multiply phases together, the phase angles add:

$$e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$$

Consequently, multiplying a phase with its conjugate always gives 1:

$$e^{i\alpha}\overline{e^{i\alpha}} = e^{i\alpha}e^{-i\alpha} = e^{i(\alpha-\alpha)} = e^0 = 1.$$

More generally, we can always get the magnitude of a complex number by multiplying with it's conjugate and taking the square root:

$$\sqrt{\lambda\bar{\lambda}} = \sqrt{(re^{i\alpha})(re^{-i\alpha})} = \sqrt{r^2e^0} = r$$

As a consequence, whenever we have $\lambda = re^{i\alpha}$ and $|\lambda| = 1$, we can conclude $r = 1$ so $\lambda = e^{i\alpha}$. In other words, $|\lambda| = 1$ if and only if λ is a phase.

Getting the angle out is a bit trickier. For this we use a trigonometric function called **arg**, which for $\lambda \neq 0$ is defined (somewhat circularly) as the unique angle α such that $\lambda = re^{i\alpha}$. If $\lambda = a + ib$ for $a > 0$, $\arg(\lambda) = \arctan(\frac{b}{a})$. This obviously won't work when a is zero, and needs a bit of tweaking when a is negative. Hence, the full definition of arg needs a case distinction, which we'll leave as an exercise.

Exercise 1:

a) Give a full definition for arg for all non-zero complex numbers, using (2) as a guide.

b) Write $\cos\alpha$ and $\sin\alpha$ in terms of complex phases. Hint: remember that $\cos -\alpha = \cos\alpha$ while $\sin -\alpha = -\sin\alpha$.

Let's also make sure we are on the same page when it comes to definitions for certain types of linear maps.

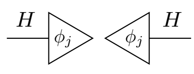
Definition 1. A linear map $M : H \to H$ is called:

- **normal** if $M^\dagger M = MM^\dagger$
- a **unitary** if $M^\dagger M = I$ and $MM^\dagger = I$
- **self-adjoint** if $M = M^\dagger$
- **positive** if $M = N^\dagger N$ for some N
- a **projector** if $M = M^\dagger = M^2$

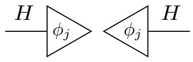
There are clearly some containments here: projectors are always positive (since $M = MM = M^\dagger M$), and positive maps are always self-adjoint (since $M^\dagger = (N^\dagger N)^\dagger = N^\dagger N = M$). Finally, everything in Definition 1 is normal. Unitaries are normal because $M^\dagger M = I = MM^\dagger$. Self-adjoint maps (and hence also positive maps and projectors) are normal because $MM^\dagger = M^2 = M^\dagger M$.

A nice feature about normal maps (and hence all the maps in Definition 1) is that they can always be **diagonalised**. That is, we can find an ONB (Ortho-Normal Basis) $\mathcal{M} = \{|\phi_j\rangle\}_j$ such that for all i, $M|\phi_j\rangle = \lambda_j|\phi_j\rangle$. The scalars λ_j and vectors $|\phi_j\rangle$ are called **eigenvalues** and **eigenvectors**, respectively, whereas $\mathcal{M}$ is called an **eigenbasis**.

Bra-ket notation (and hence string diagram notation) gives us a convenient way to write maps in diagonal form.

$$M = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j| = \sum_j \lambda_j \cdot$$


A special case is the identity map, which diagonalises with respect to *any* ONB, with eigenvalues all equal to 1:

$$I = \sum_j |\phi_j\rangle\langle\phi_j| = \sum_j$$ 

We call such a sum over an ONB a **resolution of the identity** by ONB $\{|\phi_i\rangle\}_i$.

As it turns out, all of the particular kinds of normal maps can be characterised by the types of eigenvalues they have.

Exercise 2: Let $M : H \to H$ be a linear map such that $M^\dagger M = MM^\dagger$. Then $M = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ for some sets $\{\lambda_j\}_j, \{|\phi_j\rangle\}_j$. Show that:

a) M is self-adjoint iff all $\lambda_j \in \mathbb{R}$

b) M is positive iff all $\lambda_j \in \mathbb{R}_{\geq 0}$

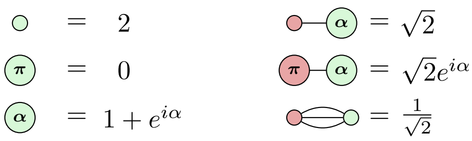
c) M is a projector iff all $\lambda_j \in \{0, 1\}$

d) M is a unitary iff all $\lambda_j \in U(1)$

where $\mathbb{R}_{\geq 0}$ is the set of all real numbers ≥ 0 and $U(1) := \{e^{i\alpha} \,|\, \alpha \in [0, 2\pi)\}$ is the set of all **phases**.

One thing you might notice is what happens when a map is unitary *and* self-adjoint: it's eigenvalues are all in $U(1) \cap \mathbb{R} = \{-1, 1\}$. This is indeed what happens with the *Pauli maps*, which as we'll see later, have many nice properties.

In the lectures we often ignore scalar factors in ZX-diagrams. We can however represent any scalar we want with a ZX-diagram. For instance, we have:



(3)

(can you see why?)

Exercise 3: By combining the diagrams from (3), find a ZX-diagram to represent the following scalar values z:

a) $z = -1$.

b) $z = e^{i\theta}$ for any θ.

c) $z = \frac{1}{2}$.

d) $z = \cos\theta$ for any value θ.

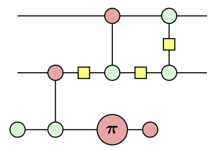
e) Find a general description or algorithm to construct the ZX-diagram for any complex number z.

The exercises continue in the *Jupyter notebook* (the file with the ipynb extension). This is a more visual programming environment for Python. To use it, make sure you have Python 3.9 or newer installed, and then install Jupyter by running `pip install notebook`. (if you are a Windows user, open a command-line first by pressing Windows+R, and then typing `cmd`. If it tells you that it can't find the program `pip`, then run `python -m pip install notebook`. If that also doesn't work, reboot your computer and try again, and otherwise ask someone tech-savvy.) For these exercises you will also need `sympy` installed, so also run `pip install sympy`.

Once you have Jupyter installed, you open the interface by typing in a command-line `jupyter notebook`. In that interface you can find the file and open it.

Gdansk Summer School
Day 2, Picturing Quantum Weirdness 2023

Exercise 1: Using ZX-calculus rewrites, help the poor trapped π phase find it's way to an exit (i.e. an output).



Note that it might be leaving with friends.

Exercise 2: There is a quantum information protocol that is closely related to teleportation, which is known as **dense coding**. Whereas quantum teleportation send classical information in order to transport quantum information, in dense coding we send a quantum state in order to transport *classical* information. The dense coding protocol works as follows:

1. Alice and Bob start with a shared Bell state.
2. Now Alice picks the two classical bits $a, b \in \{0, 1\}$ she wants to send to Bob.
3. Alice applies the following operation to her part of the state: $X(a\pi)Z(b\pi)$.
4. Then Alice sends her part of the quantum state to Bob.
5. Now that Bob has access to both quantum states, he performs a CNOT between them and a Hadamard to one of them.
6. Bob now has the quantum state $|a, b\rangle$, and so performing a measurement will give him the bits a and b.

Write this protocol as a ZX-diagram and prove using the ZX-calculus that Bob indeed ends up with the state $|a, b\rangle$. You don't have to include the measurements on Bob's side.

The reason this protocol is special is that we are sending just one quantum bit, and we are getting 2 regular bits out of it. Hence why this is called 'dense' coding.

Exercise 3 (20 points): Prove using the ZX-calculus that we can get rid of Hadamard self-loops at the cost of acquiring a phase:

(1)

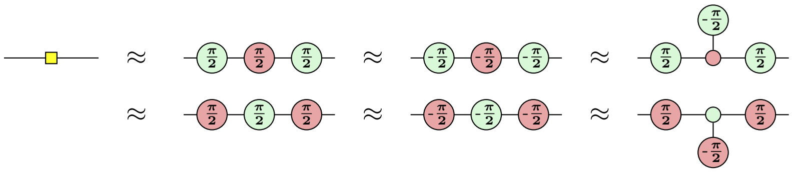
Hint: Decompose the Hadamard into spiders with a $\frac{\pi}{2}$ phase.

Exercise 3 (10 points): There are two ways in which we can write the eigenstate $|i\rangle := \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ of the Pauli operator Y (up to global phase). Namely, as $|i\rangle = R_X(-\frac{\pi}{2})|0\rangle$ or as $|i\rangle = R_Z(\frac{\pi}{2})|+\rangle$. Prove this equivalence using the ZX-calculus:

(2)

Note that an analogous equation holds for $|-i\rangle := \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$, which boils down to flipping the signs of the phases in the spiders.

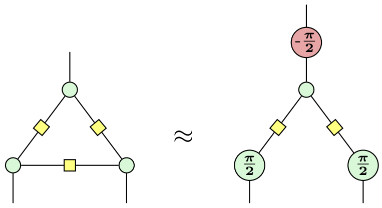
Exercise 3: The Euler decomposition from the lecture is just one possible way to write the Hadamard in terms of spiders. There is in fact an entire family of representations that will also be useful to note:



(3)

Prove that all the equations of (3) hold in the ZX-calculus, by using the top-left decomposition and the other rewrite rules of the ZX-calculus we have seen so far.

Exercise 4: Prove the following lemma using the rules of the ZX-calculus, which is the base-case of **local complementation**, a rule we will see in the coming days.



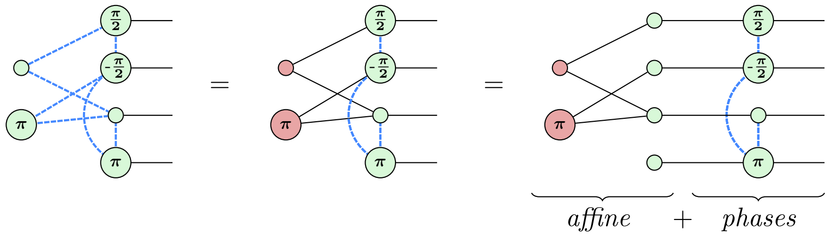
(4)

Hint: Push the top Hadamards up and decompose the middle Hadamard using one of Eq. (3) to reveal a place where you can apply strong complementarity.

Gdansk Summer School

Day 3, Picturing Quantum Weirdness 2023

In the lectures we saw that using local complementation and pivoting we can reduce any ZX-diagram to a graph-like form where the only interior spiders have a 0 or π phase and are connected only to boundary spiders. We call this the **Affine with Phases form** (AP form). For example:



(1)

This form corresponds in a direct way to a quantum state (or map) defined in terms of:

1. an **affine** subspace, defined by a system of linear equations
2. a **phase** polynomial

For example, the state above is equal, up to a scalar, to:

$$\sum_{\vec{x}, A\vec{x}=\vec{b}} e^{i\pi\cdot\phi}|\vec{x}\rangle \quad \text{where} \quad \begin{cases} A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \phi = \frac{1}{2}x_1 - \frac{1}{2}x_2 + x_4 + x_1x_2 + x_2x_4 + x_3x_4 \end{cases} \tag{2}$$

We are going to be using the AP-form in this series of exercises to argue that the ZX-calculus rewrite rules are *complete* for Clifford diagrams.

While the AP-form is a useful, compact form for Clifford ZX-diagrams, it would be *more* useful if it was unique! This is where the *reduced* AP-form comes in.

Definition 1. A diagram in AP-form defined by the triple $A, \vec{b}, \phi$ is in *reduced AP-form* if it is 0 or it is non-zero and:

1. A is in reduced row echelon form (RREF) with no zero rows,
2. ϕ only contains free variables from the system $A\vec{x} = \vec{b}$.
3. all the coefficients of ϕ are in the interval $(-1, 1]$.

Recall that RREF is the form of a matrix we get when we apply the full Gauss-Jordan reduction algorithm to a matrix. For a matrix A in RREF we call the first non-zero element in a row a **pivot** (not to be confused with the pivoting we did in the lectures), and the column this element occurs on a **pivot column**. The variable x_i is called a *free variable* if the i-th column of A is *not* a pivot column. Otherwise we call x_i a *bound variable*. Recall that these names come from the fact that if you want to find a solution to A, i.e. a vector $\vec{x}$ such that $A\vec{x} = \vec{0}$, then we can only choose the free variables. The bound variables are, well, bound by the equation $A\vec{x} = \vec{0}$.

The utility of looking at the reduced AP-form is the following theorem.

Theorem 2. For any non-zero state $|\psi\rangle$, there is at most one triple $(A, \vec{b}, \phi)$ satisfying the conditions of Definition 1 such that:

$$|\psi\rangle \approx \sum_{\vec{x}, A\vec{x}=\vec{b}} e^{i\pi\cdot\phi}|\vec{x}\rangle$$

Proof. Since $|\psi\rangle \neq 0$, the set $\mathcal{A} = \{\vec{x}|A\vec{x} = \vec{b}\}$ is non-empty. Hence, there is a unique system of equations in RREF that define $\mathcal{A}$. From this it follows that A and $\vec{b}$ are uniquely fixed. Now, for any assignment $\{x_{i_1} := c_1, \ldots, x_{i_k} := c_k\}$ of free variables, there exists $|\vec{x}\rangle \in \mathcal{A}$ such that $x_{i_\mu} = c_\mu$. Hence:

$$\langle\vec{x}|\psi\rangle = \lambda e^{i\pi\phi(c_1,\ldots,c_k)}$$

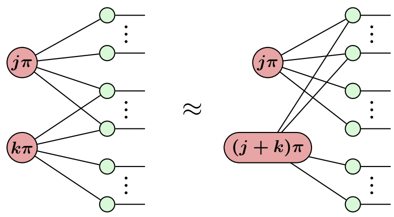
for some fixed constant $\lambda \neq 0$. From this it follows that, by inspection of $|\psi\rangle$, we can determine the value of ϕ at all inputs $(c_1, \ldots, c_k) \in \mathbb{F}_2^k$, modulo 2. This is enough to compute each of the coefficients of ϕ, modulo 2. (Q: How?) Since we require the coefficients of ϕ to be in the interval $(-1, 1]$, we can drop the "modulo 2" and conclude that ϕ is uniquely fixed by $|\psi\rangle$. $\square$

So, if we can show that the ZX-calculus lets us not only transform any Clifford ZX-diagram to AP-form, but actually all the way to *reduced* AP-form, we can appeal to Theorem 2 to prove completeness.

Exercise 1: Convince yourself that if we can reduce Clifford ZX-diagrams to a unique normal form using the rewrite rules that this means we have proven completeness.

Okay, so let's get to work showing we can reduce to reduced AP-form.

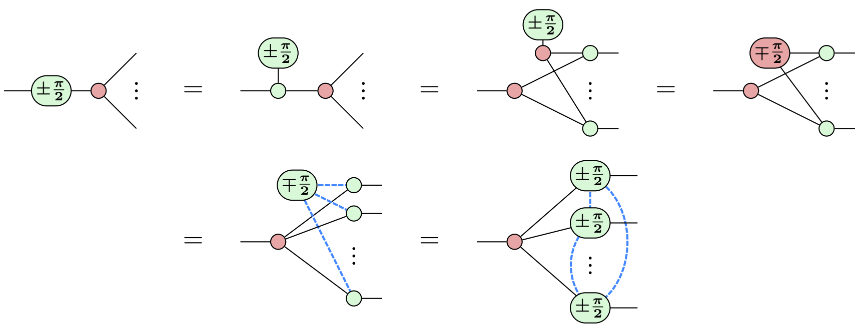
Exercise 2: The X-spiders in a ZX-diagram in AP-form correspond to rows in the linear system $A\vec{x} = \vec{b}$. Show that we can perform primitive row operations, i.e. we are able to "add" one spider to another:



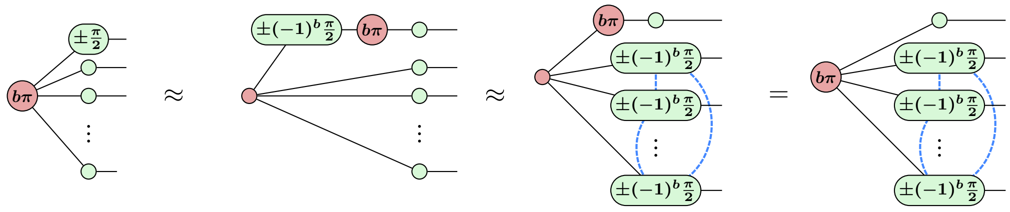
Show using this rule that we can always translate a ZX-diagram in AP-form to one with biadjacency matrix in RREF.
Hint: Strong complementarity is your friend. You will need to unfuse the phase first in order to use it though.

Now that we can reduce the matrix to RREF, we need to show that we can rewrite the phase function ϕ so that it only involves the free variables. The Z-spiders in an AP-form correspond to variables. If the biadjacency matrix is in RREF, we can recognise the bound variables as those Z-spiders connected to *exactly one* X-spider (Q: Why?). Let's call those *bound Z-spiders*. Our goal now is to remove any bound variables from the phase polynomial part of the AP-form, or graphically, to remove any phases or Hadamard edges from bound Z-spiders.

For example, we can remove a $\pm\pi/2$ phase from a bound Z-spider as follows. First, note we can push a $\pm\pi/2$ phase through a X-spider with no phase, resulting in a totally connected graph of $\pm\pi/2$ Z-spiders on the outputs:



Then, a bound Z-spider in the AP-form diagram is connected to exactly one X-spider with a phase of 0 or π. So, we can move the $\pm\pi/2$ phase off of the bound Z-spider as follows:



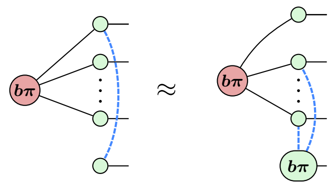
(3)

Exercise 3: Show that similarly π phases can be pushed away from the spiders corresponding to bound variables.

Okay, so now we can remove all the phases. It remains to remove the CZs the bound variables occur in.

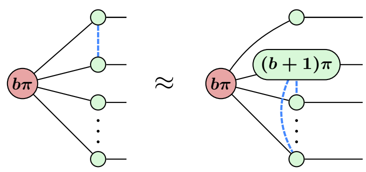
Exercise 4:

a) Prove that we can remove Hadamard edges from a pivot to a spider outside its parity set:

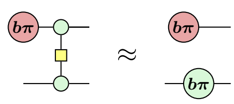


Hint: First unfuse the $b\pi$ phase and push it past the CZ. Then apply strong complementarity between the X-spider and the top Z-spider of the CZ.

b) Prove that similarly, we can also remove Hadamard edges from a pivot to a spider *inside* its parity set:



Note that the previous phase-removal and edge-removal rewrites also apply when the pivot spider is the only spider in its parity set, but then these rewrites become quite trivial:



Exercise 5: Combine the previous results into a procedure for reducing an AP-form diagram to reduced form. Use this to argue that the ZX-calculus is hence complete.

Gdansk Summer School
Day 4, Picturing Quantum Weirdness 2023

In the lectures we saw that we could go from the unitaries Hadamard, CNOT and Hadamard gates to a path-sum expression. We can also use this reasoning backwards to find circuits for unitaries we want to implement and for which we have a path-sum description. That is what we will do in this exercise.

One of the most famous unitaries is the **Quantum Fourier Transform** (QFT). This unitary is for instance the magic trick that makes Shor's algorithm work. In general, for a d-dimensional space $\mathbb{C}^d$ we define its Fourier transform as follows. Let $|0\rangle, \ldots, |d-1\rangle$ be the standard basis states. Then

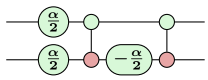
$$QFT \ :: \ |x\rangle \mapsto \frac{1}{\sqrt{d}} \sum_{y=0}^{d-1} e^{\frac{2\pi i}{d} xy} |y\rangle.$$

For instance, when $d = 2$ we get $|x\rangle \mapsto \frac{1}{\sqrt{2}} \sum_y e^{i\pi xy} |y\rangle$, which is exactly the Hadamard. We will be interested in the case where $d = 2^n$, so that the space corresponds to n qubits. We should read the expression xy then as multiplying the numbers $x, y \in \{0, \ldots, 2^n - 1\}$, *not* as taking the inner product of two bitstrings. But since we do like thinking about bitstrings, let's define a translation. For a bitstring $\vec{x} \in \mathbb{F}_2^n$, define $b(\vec{x}) := 2^{n-1}x_1 + 2^{n-2}x_2 + \cdots + 2^0 x_n$. Then we can write the n-qubit QFT as follows:

$$|\vec{x}\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{\vec{y} \in \mathbb{F}_2^n} e^{\frac{2\pi i}{2^n} b(\vec{x}) b(\vec{y})} |\vec{y}\rangle. \tag{1}$$

Exercise 1:

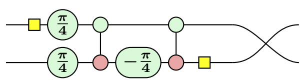
a) To compile the QFT unitary to a quantum circuit, we will need to know first how to compile *controlled phase* gates. These are unitaries that act like $|x_1, x_2\rangle \mapsto e^{i\alpha x_1 \cdot x_2} |x_1, x_2\rangle$. Using the path-sum approach and the identity $x \cdot y = \frac{1}{2}(x + y - x \oplus y)$ show that the following circuit implements a controlled phase gate:



(2)

b) For $n = 2$, expand the definition of b in Eq. (1) to write the phase polynomial as a product of $e^{i\alpha x_k y_l}$ terms. Argue that the $x_1 y_1$ term can be removed without changing the resulting map.

c) Using the path-sum approach, show that the following circuit implements the 2-qubit QFT, by verifying that it is equal to the expression you derived above:



(3)

Note that you might need to 'rename' the variables y you are getting out of an application of a Hadamard gate to make sure it matches the expression you had above.

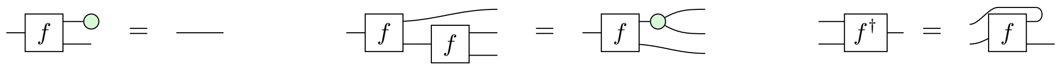
d) Now for $n = 3$, again expand the definition of b in Eq. (1) to write the phase polynomial as a product of $e^{i\alpha x_k y_l}$ terms and remove the terms that you can remove. Construct a circuit to implement the 3-qubit QFT.

e) Give a recipe for constructing the n-qubit QFT for any n.

Now in the following exercises we will look at Pauli gadgets in a slightly more abstract way, motivating why they work the way they do.

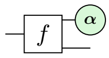
First, let's define the following.

Definition 1. Let $f : A \to \mathbb{C}^2 \otimes A$ be a linear map. We say it is a *measure-box* when it satisfies the following identities:

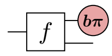


We call this map a measure-box because it allows us to define a 'von Neumann' measurement (with 2 outcomes) on system A (these are 'incomplete' measurements that only give you partial information about the resulting state). Note that, while the first output wire is always a 2D space, the system A which appears on the input wire and the second output wire can be more general. In the following exercises A will usually be a tensor product of multiple qubits.

Exercise 2: Let f be a measure-box.

a) Show that the following is unitary for any choice of α:  . You need to use here that taking the adjoint of spiders flips the phase.

b) Show that the following is a projector for $b \in \{0, 1\}$:

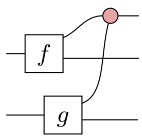
$$f_b := \frac{1}{\sqrt{2}} \cdot$$ 

Recall that a projector P is a map satisfying $P^2 = P^\dagger P = P$.

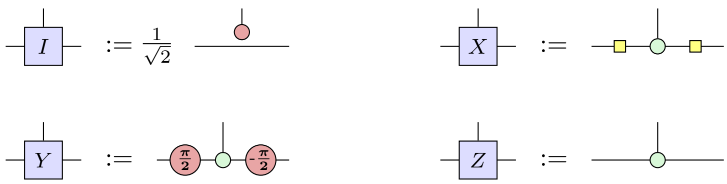
c) Show that the above projectors for $b = 0$ and $b = 1$ are in fact *orthogonal* projectors, i.e. that $f_0 \circ f_1 = f_1 \circ f_0 = 0$.

d) Show that f_0 and f_1 form a *resolution of the identity*, namely $f_0 + f_1 = I$.

Exercise 3: Let $f : A \to \mathbb{C}^2 \otimes A$ and $g : B \to \mathbb{C}^2 \otimes B$ be two measure-boxes. Show that the following combined process is also a measure-box (up to a scalar) from $A \otimes B$ to $\mathbb{C}^2 \otimes A \otimes B$:



So now that we have these abstract boxes with nice properties that can be combined together, let's make it more concrete. We define the *Pauli boxes* as follows:



We've drawn the wires coming out at the top, since it will turn out not to matter if we treat them as inputs or outputs, due to the next exercise. But for now you can treat them as being output wires.

Exercise 4:

a) Show that each of the Pauli boxes is a measure-box. *Hint*: you can prove the result for X, Y, and Z in one go with a good choice of lemma.

b) Show that if we plug $-\!\!(\pi)$ into each of the Pauli boxes' top wires that the result is the Pauli that it is named after.

Okay, so the Pauli boxes are measure-boxes that give back the Pauli's when we plug in the right thing to their top wire. Using the construction of Exercise 3 we can then combine these Pauli boxes into larger measure-boxes, so that for any Pauli string $P_1 \otimes P_2 \otimes \cdots \otimes P_n$ we can build an associated measure-box. Then, by plugging in a $-\!\!(\alpha)$ it becomes a unitary by Exercise 2a).

Exercise 5: Show that the unitaries we get by combining Pauli boxes using the constructions of Exercise 3 and then 2a), are Pauli exponentials, by showing that the diagrams agree with those found in the lecture.

When we have two *different* Pauli's $P, Q \in \{X, Y, Z\}$, they always *anti-commute*: $PQ = -QP$ (two copies of the same Pauli of course commute with one another). This means that when we have Pauli strings $\vec{P}$ and $\vec{Q}$, they will either commute or anti-commute depending on how many of its factors commute or anti-commute. This (lack of) commutation translates onto the associated Pauli exponentials as well.

Exercise 6:

a) Prove diagrammatically that the Pauli exponentials $ZZ(\alpha)$ and $XX(\beta)$ commute for any value of α and β.

b) Try doing the same rewrite strategy with the (non-commuting) $ZZ(\alpha)$ and $ZX(\beta)$. What goes wrong?

c) Describe the condition on general Pauli strings that is required to allow you to commute the associated Pauli exponentials past each other.
